Cooperation in aspiration-based N-person prisoner's dilemmas

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We propose a mathematical model of the N-person prisoner's dilemma game played by a continuous population of agents with a time-dependent aspiration level. The model—a system of differential equations—takes into account the evolution of the aspiration level and of the mean frequency of the cooperators in the population. The dependence of the asymptotic level of cooperation on the individual payoffs and on the transition rates determining the agent's reaction to the received payoffs is studied. In general the existence and the magnitude of the asymptotic level of cooperation depends on N, the payoffs and the transition rates, and decreases with increasing N.

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I. INTRODUCTION

Evolution of cooperation is one of the most important social phenomena studied extensively in social, economic, biological, and other contexts. The qualitative measures of cooperation are important characteristics of social groups.

In many scientific disciplines, which investigate macroscopic systems of entities with microscopic interactions, the theoretical description of such systems is based on the binary contests. The multiplayer game theory is much more complicated than the theory for two-player games. However, the multiplayer interactions are important in the biological world, in the economy, as well as in the theoretical description of human behavior. The payoffs from such multiplayer contests can be different from binary ones. In the former case all the players take part in one simultaneous contest. In symmetric contests the payoff of each player depends only on the number of other players who play each of the available strategies. Such multiplayer contests can significantly change the time evolution and the equilibrium properties of the stationary states of the population.

One of the most popular and important for applications games is the prisoner's dilemma (PD) game, considered as a paradigm for the evolution of cooperation in the social, biological, and other contexts, cf. e.g., [1,2] for recent reviews. Majority of the work on the PD game is focused on the two-person contests. The multiperson PD has been also recently studied in various contexts (cf. e.g., [3–13], and references cited therein).

In the case of the continuous populations of players playing the standard two-person PD, without any additional mechanism which would facilitate the cooperation, it is expected that in the long run the only stable outcome of the evolution is the total defection. In [14] an aspiration-based model of cooperation in a continuous system of agents matched to play a two-person PD at any instant of time has been proposed. The aspiration level was a global variable, updated on the basis of a function of the actual frequency of cooperators. Introduction of such reinforced-learning mechanism enabled partial cooperation in the long run.

We develop the idea of the continuous system of agents with variable aspiration, generalizing the interaction part in order to study systematically the dependence of the cooperation level on the number N of agents which take part in the contest.

We formulate a general model for arbitrary group size N, and determine the stationary points of the resulting system of ordinary differential equations. We demonstrate how the level of (partial) cooperation changes with the increasing size of the group for a wide range of parameters determining the strategy choice of the agents, and discuss—using analytical tools—examples of multiple equilibria.

II. MODEL

We define the N-person prisoner's dilemma (N-PD) as a one stage game in which N identical players simultaneously choose one of two actions: cooperate (C) or defect (D), and obtain the payoffs which depend only on their choice of the action and on the number of other players who play C and D. The structure of the payoffs should preserve the essence of the dilemma (see below).

We denote by R_k the payoff of the player who chooses C in the N-PD game in which k opponents $(0 \leq k < N)$ defect, and $N-k-1$ opponents cooperate, and by T_k the payoff of the player who chooses D if $k-1$ opponents $(0 < k \leq N)$ play D, and $N-k$ opponents play C. In particular, for $N=2$, with the identification $R_0=R$, $R_1=S$, $T_1=T$, $T_2=P$, and ordering $T > R > P > S$ we obtain the usual two-person PD, with the payoff matrix

	C	D
C	R	S
D	T	P

which will be abbreviated by $[T,R,P,S]$.

The payoffs in the N-PD must satisfy the general requirements of the prisoner's dilemma. In particular, the highest profit belongs to the defector who plays with $N-1$ cooperators, and is greater than that of each of two defectors playing with $N-2$ cooperators, etc., whereas the smallest payoff is that of the cooperator who plays with $N-1$ defectors. More in detail, one assumes, cf. [13,15]:

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(1) For any fixed number of defectors k, the agent who chooses D is better off than that who chooses C: $T_k > R_{k-1}$, $0 < k \leq N$.

(2) The average payoff of the group of N players increases if the number of the cooperators increases. Thus, with k D players and $N-k$ C players, the average payoff, defined as $P_N(k) = kT_k + (N-k)R_k$, satisfies for all N and $0 \leq k < N$ the inequality $P_N(k) > P_N(k+1)$.

(3) The individual payoffs should increase for both defectors and cooperators as the number of defectors in the group decreases: $R_k < R_{k-1}$, $0 \leq k < N$, $T_k < T_{k-1}$, $0 < k \leq N$.

There are many ways of satisfying the above requirements for the individual payoffs of the players in the N-PD. In this paper the payoffs of the players in the N-PD game in which k players defect and $N-k$ players cooperate are defined as a sum of 2-PD contests as follows: For each C player the payoff from the N-PD game with $N-k-1$ cooperators and k defectors is

$$R_k = (N-k-1)R + kS, \quad 0 \leq k < N. \tag{1}$$

For each D player the payoff from the N-PD game with $N-k$ cooperators and $k-1$ defectors is defined as

$$T_k = (N-k)T + (k-1)P, \quad 1 \leq k \leq N. \tag{2}$$

We refer to these payoffs as *accumulated* payoffs. It can be proved by straightforward calculations that the above choice of the payoffs satisfies the consistency conditions (1)–(3), and with relevant normalization is equivalent to the Public Good game [7–9]. Note that for $N > 2$ the above definitions do not determine uniquely the ordering of all the individual payoffs, and different types of the N-PD can be considered, as will be discussed below.

We assume the evolutionary scenario in which the continuum of players is matched to play, at any instant of time, the N-PD. Let $\mu = \mu(t)$ be the proportion of the cooperators in the system. The payoff of a randomly chosen player is defined as

$$\Pi(\mu) = \mu \Pi_C + (1-\mu)\Pi_D, \tag{3}$$

where Π_C and Π_D are, respectively, the mean payoffs of a cooperator and a defector,

$$\Pi_C = \sum_{k=0}^{k=N-1} P_k^N(\mu) R_k = (N-1)[\mu R + (1-\mu)S], \tag{4}$$

$$\Pi_D = \sum_{k=0}^{k=N-1} P_k^N(\mu) T_{k+1} = (N-1)[\mu T + (1-\mu)P], \tag{5}$$

where $P_k^N(\mu) = \binom{N-1}{k} \mu^{N-1-k}(1-\mu)^k$.

We model the evolution of the system properly generalizing [14]. The population of noncognitive actors is assumed to have a global time-dependent aspiration level $\alpha(t)$. If the payoff of an agent is lower than the actual aspiration level, then the agent feels dissatisfied, and switches her action at a rate that depends on the difference between the payoff (R_k or T_k) from the actual strategy and the aspiration level $\alpha(t)$. The transition rate is defined by a function $f(\chi)$, which determines how fast the agents react when dissatisfied.

Evolution equation for the frequency of cooperators for the N-PD reads

$$\dot{\mu} = -\mu \left\{ \sum_{k=0}^{k=N-1} P_k^N(\mu) f(\alpha - R_k) \right\} + (1-\mu) \left\{ \sum_{k=0}^{k=N-1} P_k^N(\mu) f(\alpha - T_{k+1}) \right\}, \tag{6}$$

where

$$f(\chi) \begin{cases} =0, & \text{if } \chi \leq 0 \\ >0, & \text{if } \chi > 0 \end{cases} \tag{7}$$

is a nondecreasing continuously differentiable on R^+ function, which gives the probability rate at which the agent who receives the payoff π changes her action when the current aspiration level is $\alpha = \pi + \chi$.

In Eq. (6) each negative term, labeled by k, corresponds to the interaction of a cooperator with k defectors and $N-1-k$ cooperators, in which the cooperator tends to switch to defection if she is dissatisfied, i.e., if the cooperator's payoff R_k is smaller than the current aspiration level $\alpha(t)$. Analogously, the positive terms account for the increase in the cooperator frequency μ from the dissatisfied defectors.

A similar learning model was considered in [16]. The authors elaborated the Bush-Mosteller model of learning. They investigated a dynamical system in which there was a feedback between the probability of playing a given strategy (C or D), and the aspiration level of the system. The probability increases if the stimulus (the normalized difference between the payoff and the current aspiration level) is positive, decreases otherwise.

Equation (6) can be rewritten in a compact form,

$$\dot{\mu} = \sum_{k=0}^{k=N-1} P_k^N(\mu) G_k(\mu), \tag{8}$$

where $G_k = -\mu f(\alpha - R_k) + (1-\mu) f(\alpha - T_{k+1})$. In particular, for the Heaviside switch function

$$f_0,(\chi) = \begin{cases} 0, & \chi \leq 0 \\ 1, & \chi > 0 \end{cases} \tag{9}$$

G_k takes only one of the four values: 0, $-\mu$, $1-2\mu$, $1-\mu$, and the right-hand side (rhs) of Eq. (8) is a polynomial of at most Nth order. This property will be used in proof of Theorem 1 and Examples 1 and 2 below.

The aspiration level can be a fixed constant or an unknown time-dependent function. In Appendix A we prove that the asymptotic cooperation level in a 3-PD is lower than that in the 2-PD if the aspiration level is constant. Below we consider the general N-PD model with time-dependent aspiration. Following [14] we assume the evolution of the aspiration function to be governed, on the same time scale, by

$$\dot{\alpha} = \Pi(\mu) - \alpha. \tag{10}$$

The system (6)–(10) is a rather complicated nonlinear two-dimensional (2D) systems of differential equations, with—in general—nondifferentiable rhs. A remarkable feature of the proposed model is that for each N the stationary points of Eqs. (6)–(10) can be found by solving the single algebraic equation

$$F(\mu,\Pi(\mu)) = 0, \quad (11)$$

where $F=F(\mu,\alpha)$ denotes the rhs of Eq. (6), and, from Eq. (3),

$$\alpha = (N-1)[\mu^2 R + \mu(1-\mu)(S+T) + (1-\mu)^2 P]. \quad (12)$$

In the following the solutions of Eq. (11) which belong to the interval (0,1) will be referred to as the (partial) cooperation levels of the system.

In the next section we find the solutions of Eq. (11) for different choices of the N-PD payoffs and switch functions f. We discuss the dependence of the stationary solutions of Eqs. (6)–(10) on these parameters of the system and on the group size N. In order to get more insight into the structure of the solutions, and to understand qualitatively the emerging phenomena, in the current section we provide rigorous results on the existence and some properties of the cooperation levels for the specific choices of the switch functions, for the N-PD with $2 \le N \le 5$. The important role in the discussion of the analytical results will be played by the systems with the Heaviside switch function. In such systems the agents react "instantaneously" to the differences between their payoffs and the actual aspiration level of the system. First we show (Theorem 1 and Examples 1 and 2) that for the Heaviside switch function the number and the magnitude of the partial cooperation levels depend on the order N of the game. More in particular, we prove that for $N=2$ and $N=3$ there is at most one partial cooperation level for any choice of the 2-PD parameters T,R,P,S, whereas for $N \ge 4$ the situation becomes more complex, and more cooperation levels are possible. Next we show (Theorem 2) that for the linear switch function, already for the 2-PD the number and the magnitudes of the partial cooperation levels depend in a nontrivial way on the rate of the strategy switching and on the payoffs of the PD.

Theorem 1. For $N=2$ and $N=3$ there exists at most one cooperation level μ^* for the systems (6)–(10) with Heaviside switch function (9) for all $T>R>P>S$.

Proof. The individual payoffs satisfy for all N the inequalities $T_1 > T_2 \cdots > T_N$, and $R_0 > R_1 > \cdots > R_N$. The numerical values of T,R,P,S determine the orderings of T_{k+1} and R_k and the analytical form of the polynomials in Eq. (8).

For $N=3$, from Eqs. (1) and (2) we obtain $R_0=2R$, $R_1=R+S$, $R_2=2S$, $T_1=2T$, $T_2=T+P$, and $T_3=2P$. There are four generic (i.e., with sharp inequalities) orderings of the above payoffs (for particular T,R,P,S some of them should be replaced by equalities), giving rise to four types of the 3-PD as follows:

$$T_1 > R_0 > T_2 > R_1 > T_3 > R_2,$$

$$T_1 > R_0 > T_2 > T_3 > R_1 > R_2,$$

$$T_1 > T_2 > R_0 > R_1 > T_3 > R_2,$$

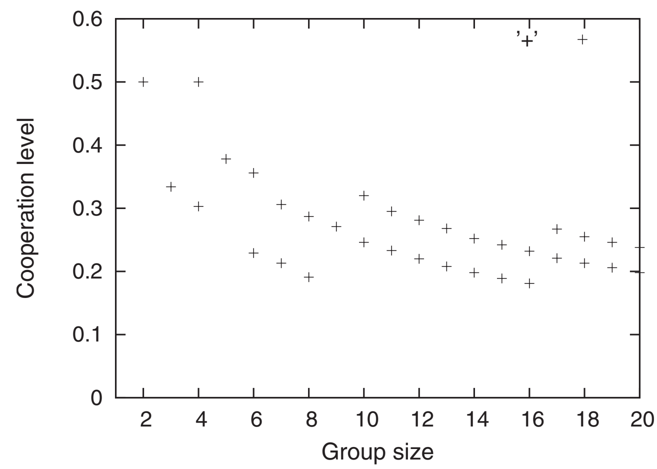

FIG. 1. Cooperation levels, $f=f_0$, $T=1.5$, $R=1$, and $S=-0.4$.

$$T_1 > T_2 > R_0 > T_3 > R_1 > R_2.$$

For each of the above orderings of R_k and T_k we proceed as follows. Let (μ^*,α^*) be a stationary state of Eqs. (6)–(10); α^* belongs to one of the intervals determined by the ordering. For the assumed interval of α^* we write down the explicit form of the function $F=F(\mu^*,\Pi(\mu^*))$ (which is here a polynomial of the third or lower degree) and find solutions $\mu^* \in (0,1)$ of Eq. (11). Then we check whether α^*, calculated from the consistency condition $\alpha^*=\Pi(\mu^*)$, belongs to the chosen interval. If positive, then μ^* is the cooperation level of the system. In Appendix B we give details of the relevant calculations for $N=3$. For $N=2$ the proof is analogous. ■

For $N \ge 4$ Theorem 1 is not valid. Below we give an example of the existence of two cooperation levels for $N=4$.

Example 1. We choose $T=1.5$, $R=1$, $P=0$, and $S=-0.4$, which gives the following ordering of the payoffs in the 4-PD:

$$T_1 > R_0 > T_2 > R_1 > T_3 > R_2 > T_4 > R_3. \quad (13)$$

The ordering divides the domain of the admissible values of α^* into intervals: $(-\infty,R_3]$, $(R_3,T_4]$, $\ldots,(R_0,T_1]$, $(T_1,+\infty)$. For α^* from each such interval $(a,b]$ we calculate, for the Heaviside switch function, the roots $\mu^* \in (0,1)$ of the polynomial defined by the rhs of Eq. (8). If the consistency condition $\alpha^*=\Pi(\mu^*) \in (a,b]$ is satisfied, then μ^* is the searched cooperation level. Subsequently applying this procedure to all the intervals determined by ordering (13) we obtain two cooperation levels. For the interval $(R_2,T_3]=(0.2,1.5]$ $\mu_1^* \approx 0.303$, $\alpha_1^* \approx 0.97$, and for the interval $(T_3,R_1]=(1.5,1.6]$ $\mu_2^*=0.5$, $\alpha_2^*=1.575$. Numerical calculations indicate the local asymptotic stability of the corresponding stationary points of Eqs. (6)–(10). For the other intervals the above consistency conditions on α are not satisfied.

In Fig. 1 we show the cooperation levels for $N=2,3,\ldots,20$ for the data of Example 1. Note two stationary points of the system for $N=4$ and for other group sizes. However, there is no regular correspondence between the order N of the game and the number of the cooperation levels. This is demonstrated in Example 2 for $N=5$.

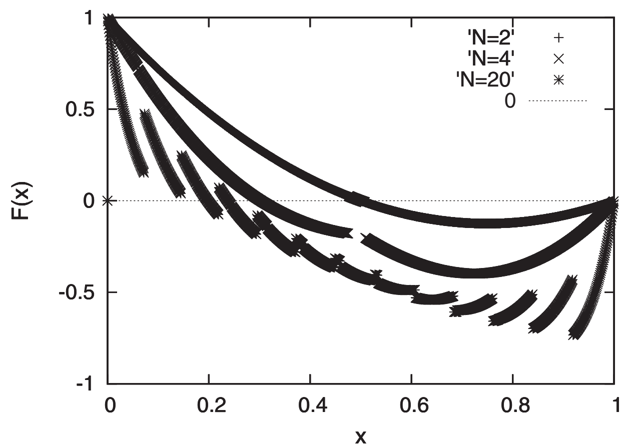

FIG. 2. Plots of F, $f=f_0$, $T=1.5$, $R=1$, and $P=0$, $S=-0.4$.

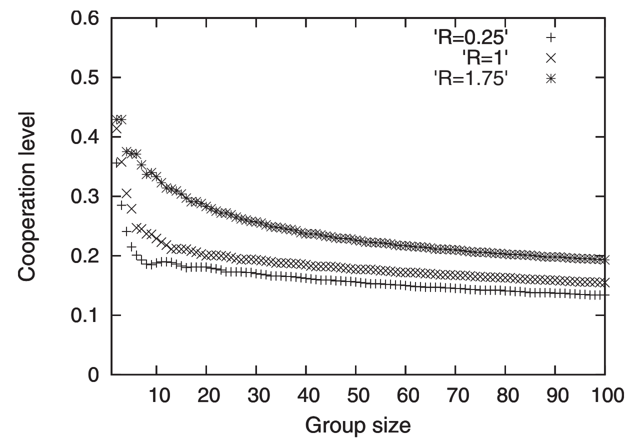

FIG. 3. Cooperation levels, $f=f_1$, $p=0.5$, $T=2$, and $S=-0.25$.

Example 2. For the 5-PD and (the same as in Example 1) the basic payoffs $T=1.5$, $R=1$, $P=0$, $S=-0.4$ we obtain the following ordering of the payoffs:

$$T_1 > T_2 > R_0 > T_3 > R_1 > T_4 > R_2 > T_5 > R_3 > R_4.$$

In Appendix C we rigorously show that there is a unique cooperation level (cf. also Fig. 1 for $N=5$).

In general, the existence of more than one cooperation level for various group sizes results from the analytical forms of the polynomials (or more complicated functions in the case of other switch functions) in Eq. (11). In Fig. 2 we present the graphs of the function F in Eq. (11) for the data from Fig. 1 for $N=2,4,20$. Note two zeros of the function F, corresponding to the relevant cooperation levels in Fig. 1 for $N=4,20$, and one zero for $N=2$. For $N=2$ the function F is continuous, For $N=4$ the plot is composed of three polynomials (F has two points of discontinuity); for larger N the number of different polynomials (and therefore discontinuities) increases.

As can be anticipated from Theorem 1 and the above examples, the results—in particular the number and the values of the asymptotic aspiration levels—depend for arbitrary N not only on the payoff matrix of the N-PD, but also on the parameters of the switch functions. For arbitrary switch functions the analytical formulas are hard to obtain. In order to get more insight into the general results of the next section, we show below how the payoffs and the parameters of the switch functions influence, already for $N=2$, the existence and the magnitude of the aspiration levels for the linear switch function

$$f_1(\chi) = \begin{cases} 0, & \chi \le 0 \\ p\chi, & 0 < \chi \le 1/p \\ 1, & \chi > 1/p. \end{cases} \tag{14}$$

More in particular we prove the following theorem.

Theorem 2. Consider the 2-PD system (6)–(10) with the switch function f_1 and the payoff matrix $[T,1,0,1-T]$, $T>1$. Then

(1) For $T \ge 2/p$, $p \in (1,2)$, there exists the unique T-independent cooperation level $\mu^* = (p-1)/p$.

(2) For $T \in (1,2]$, $T < 2/p$, there exists the unique, p-independent cooperation level $\mu^* = (2-T)/2$.

(3) For $p \ge 2$ and $T > 1$ there exists the unique cooperation level $\mu^* = 1/2$.

(4) For $0 < p \le 1$, $T \ge 2$, the cooperation level does not exist.

Theorem 2 describes the systems which depend on two parameters: the payoff parameter T and the parameter p which characterizes the speed of the strategy changes of the dissatisfied agents. Note that the payoff matrix $[T,1,0,1-T]$, $T>1$ can be interpreted as the cost-benefit 2-PD payoff matrix $[b,b-c,0,-c]$, $b>c$, with the additional normalization $b-c=1$.

Proof. The payoffs in the payoff matrix $[T,1,0,1-T] \equiv [T_1,R_0,T_2,R_1]$ determine the partition of the real axis into five intervals, out of which only $(T_2,R_0]=(0,1]$ is of interest as a domain of the admissible equilibrium aspiration levels α^* [note that here $\alpha^* = \Pi(\mu^*) = \mu^*$]. Equation (11) takes the form (after division by $1-\mu$)

$$-\mu f_1(\alpha - R_1) + (1-\mu) f_1(\alpha - T_2) = 0 \tag{15}$$

with

$$f_1(\alpha - X) = \begin{cases} p(\alpha - X), & 0 < p(\alpha - X) \le 1 \\ 1, & p(\alpha - X) > 1, \end{cases} \tag{16}$$

where $X=R_0$ or $X=T_2$. Thus, the function f_1 in Eq. (15) is equal to 1 or $p(\alpha - X)$, which implies four combinations of the consistency conditions on μ^*. Let us consider the case $f_1(\alpha^* - R_0)=1$, and $f_1(\alpha^* - T_2) = p(\alpha^* - T_2)$, which gives the consistency conditions $\alpha^* > 1/p + 1 - T$, and $\alpha^* \le 1/p$. Inserting these values of f_1 into Eq. (15) we find the solution $\mu^* = (p-1)/p$, valid under the conditions defined in the point 1 of Theorem 2. Analogously we consider the other three combinations of the values of f_1, which lead to points 2, 3, and 4 of Theorem 2. ■

Using the same technique, in Example 3 of Appendix C we calculated the cooperation level for the linear switch function f_1 with $p=1$ for $N=3$ and the payoff matrix $[T,1,0,1-T]$, $T>1$.

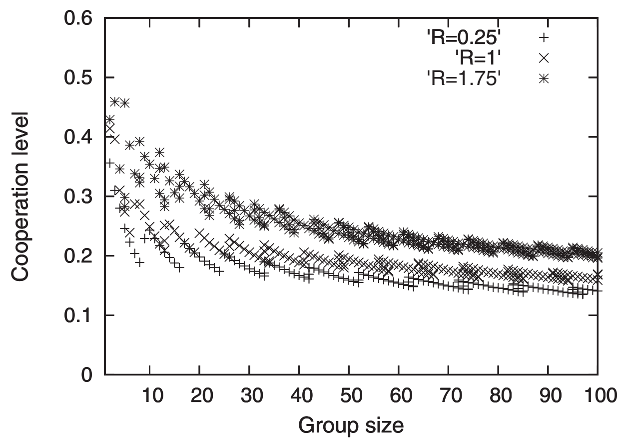

FIG. 4. Cooperation levels, $f=f_1$, $p=1$, $T=2$, and $S=-0.25$.

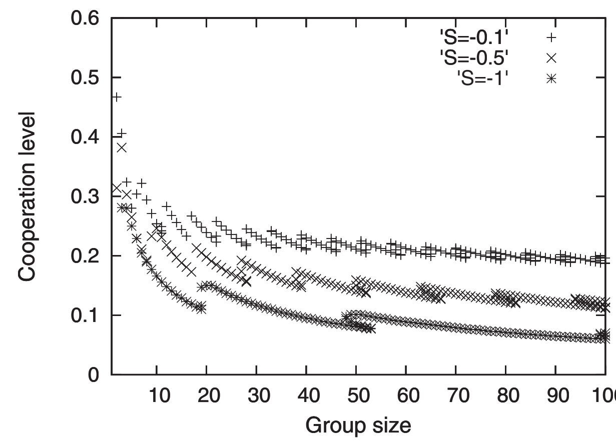

FIG. 5. Cooperation levels, $f=f_1$, $p=1$, $T=2$, and $R=1$.

The above analytical results show that already for the simple switch functions the existence and the magnitudes of the partial cooperation levels of the system are very sensitive to the variations of the payoffs and of the parameters which determine the speed of the strategy changes. In Sec. III we find the partial cooperation levels for increasing sizes of the group, various payoffs matrices, and different switch functions.

III. GENERAL RESULTS

In the calculations we choose the basic 2-PD matrix in the form $[T,R,0,S]$, with $T>R>0>S$. We solved Eq. (11) for varying payoffs T,R,S, different switch functions, and the group sizes $N=2,3,\ldots,N_{\max}$. In principle $N_{\max}$ can be arbitrarily large; we choose $N_{\max}=100$.

Apart from the functions f_0 considered above and f_1 we discuss below the results for the switch function

$$f_2(\chi)=\begin{cases}0, & \chi\leq 0\\ \dfrac{\chi^l}{v+\chi^l}, & \chi>0,\end{cases} \tag{17}$$

and briefly comment on results obtained for some other switch functions.

Mathematical treatment of the dynamical system (6)–(10) (e.g., the sufficient conditions on the switch function f which would guarantee the existence of a unique asymptotic state with partial cooperation for arbitrary N) is a formidable task. Using the methods from [14] we found for $N=3$ sufficient conditions for the switch function, analogous to those found in the cited paper for $N=2$. For arbitrary N we present below solutions of Eq. (11) obtained for different switch functions and payoffs.

In general, for weakly increasing switch functions the cooperation level μ^* decreases with the increasing group size N. However, this dependence is not necessarily monotonic, as can be seen from Fig. 3, which shows the dependence of the partial cooperation level μ^* on the reward payoff R for the switch function f_1 with $p=0.5$. Note that, as expected, for fixed N the increase in the reward R implies the increase of the cooperation level.

For comparison, in Fig. 4 we present analogous plots for the switch function f_1 with $p=1$. We notice the emergence of several cooperation levels for various group sizes, for reasons discussed rigorously in the previous section.

In Fig. 5 we show the dependence of the cooperation level on the payoff S. The cooperation level decreases when S decreases.

Similar regularities were observed for other switch functions. As an illustration, in Fig. 6 we present the dependence of the cooperation level μ^* on the reward payoff R for the switch function f_2. The increase in R furthers cooperation for all N.

We also carried out calculations for other payoffs, e.g., [3,5,1,0], used in the celebrated Axelrod's tournament [17], cf. Fig. 7, and for other switch functions, e.g., $f_3(\chi)=\mathrm{th}(q\chi)$, $\chi>0$, and $f_4(\chi)=1-\exp(-r\chi^s)$, $\chi>0$, with qualitatively similar results.

In this paper we assumed that the relaxation of the aspiration level, Eq. (10), and of the proportion of the cooperators, Eq. (6), occur on the same time scale. Alternatively, for fast enough relaxation of $\alpha(t)$, the aspiration level would be equal to the average payoff, and the system would be described by Eq. (6). When Eqs. (6)–(10) are solved, their stationary points should be the same in both cases; however their stability might differ.

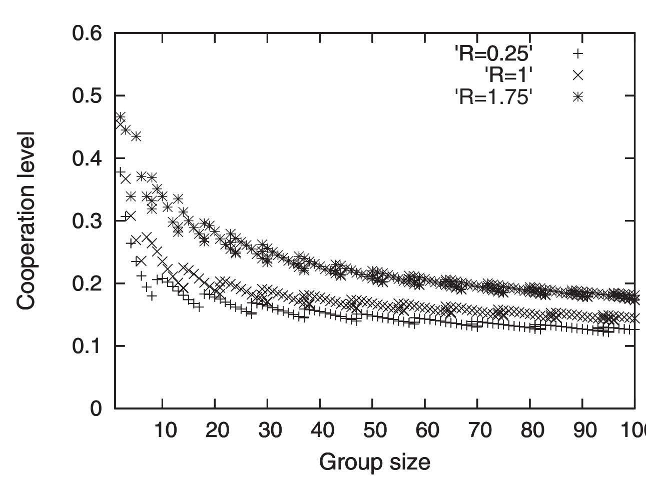

FIG. 6. Cooperation levels, $f=f_2$, $l=1$, $v=1$, $T=2$, and $S=-0.25$.

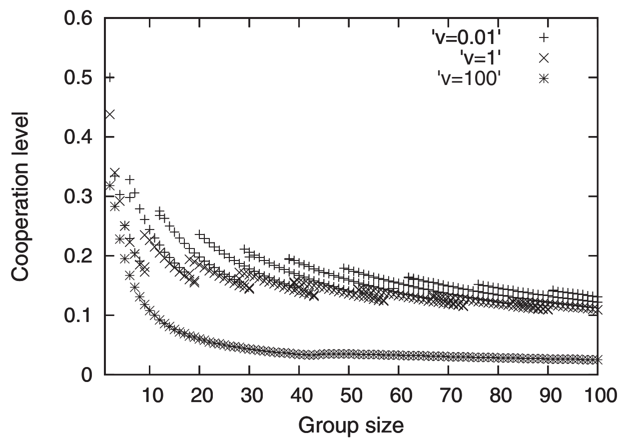

FIG. 7. Cooperation levels, $f=f_2$, $l=1$, $T=5$, $R=3$, $P=1$, and $S=0$.

Alternatively to Eqs. (1) and (2) we used another definition of the individual payoffs (referred to as *averaged* payoffs): $\tilde{T}_k=T_k/(N-1)$, $\tilde{R}_k=R_k/(N-1)$. All the averaged payoffs are bounded from below by S and from above by T. Except the four marginal payoffs T_1,R_0,T_N,R_{N-1}, the averaged payoffs depend on N. For the averaged payoffs, R_k and T_k in Eq. (6) are replaced by $\tilde{T}_k=T_k/(N-1)$, $\tilde{R}_k=R_k/(N-1)$, the payoff of a randomly chosen player is defined as $\tilde{\Pi}=\Pi/(N-1)$, and does not depend on the group size. Qualitative conclusions are the same as for the accumulative payoffs and will be not presented here. Note that in contrast to the accumulative payoffs, for which the asymptotic aspiration level α linearly increases with N [cf. Eq. (12)], the asymptotic aspiration level for the averaged payoffs does not depend on N.

IV. DISCUSSION

We proposed a hierarchy of the aspiration-based reinforced learning models of continuous system of players matched to play the N-PD games. The learning occurs through endogenous global aspiration, updated on the basis of the mean population payoff. The hierarchy is defined for arbitrary group size, which enables to study the dependence of the behavior of the population on the group size. The key role is played by the switch function, which determines the speed of the strategy changes in the system.

An important feature of the model is that it allows to study analytically the dependence of the asymptotic proportion of the cooperators in the system of agents playing the N-PD on the payoff structure of the game, and on the characteristics of the function which describes sensitivity of the agents to the differences between their payoffs and their expectations, the latter measured by a global variable—the dynamically changing aspiration.

The results capture the intuition that in general cooperation seems to be more difficult when more players participate in the contests described by the presented generalization of the two-person aspiration-based PD game to arbitrary size group. Correspondingly, the aspirations of the players become lower in larger groups. For slowly increasing switch functions the decrease is in general more regular, and the partial cooperation level is unique. For steeper switch functions the structure of the relevant phase space is more complicated, with multiple locally stable stationary points. In that case the asymptotic cooperation level depends on the initial distribution of the strategies. The effect depends on the numerical values of the parameters of the switch functions and on the payoffs of the game.

There are various interesting open questions related to the considered model. As future research subjects we mention models in which the information possessed by the agents is not homogeneous, in particular in which agents with different strategies could have different switch functions and different aspiration levels, and their evolution would depend on the mean payoffs of the corresponding groups. For the individual agent-based aspiration levels, cellular automata models seem to be an interesting alternative. The reaction of agents could depend on the size of the group—people often react differently in larger groups than in the smaller ones. The influence of the synergy effects and discounting (cf. [9]) could also be of interest in the frame of the proposed model.

Other types of social dilemmas, e.g., the snow-drift or stag-hunt games could also be studied in the proposed evolutionary setting. Recently the N-person stag-hunt dilemmas were studied from the evolutionary perspective in [10]. In particular the authors proved that the number of the stationary points of the considered replicator-type continuous dynamics (without learning) depends on the group size N.

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APPENDIX A

Lemma 1. Consider the system with the cooperation level defined by Eq. (6) for $N=2,3$, with the averaged payoffs $\tilde{T}_k=T_k/(N-1)$, $\tilde{R}_k=R_k/(N-1)$, cf. Eqs. (1) and (2), and a fixed aspiration level α. If $\max\{(R+S)/2,P\}<\alpha\leq\min\{(T+P)/2,R\}$, then the asymptotic partial cooperation levels for $N=2$ and $N=3$ satisfy

$$\bar{\mu}_2>\bar{\mu}_3 \quad (A1)$$

for arbitrary switch function f.

Proof. For $N=2$ and 3 Eq. (6) reads, respectively,

$$\dot{\mu}=-\mu^2 f(\alpha-R)-\mu(1-\mu)f(\alpha-S)+\mu(1-\mu)f(\alpha-T)+(1-\mu)^2 f(\alpha-P), \quad (A2)$$

$$\dot{\mu}=-\mu^3 f(\alpha-R)-2\mu^2(1-\mu)f(\alpha-R_1)-\mu(1-\mu)^2 f(\alpha-S)+\mu^2(1-\mu)f(\alpha-T_1)+2\mu(1-\mu)^2 f(\alpha-T_2)+(1-\mu)^3 f(\alpha-P), \quad (A3)$$

where $R_1=(R+S)/2$, $T_1=T$, and $T_2=(T+P)/2$. For the 2-PD we calculate from Eq. (A2)

$$\bar{\mu}_2 = \frac{b}{a+b}, \tag{A4}$$

and for the 3-PD, from Eq. (A3),

$$\bar{\mu}_3^{\mp} = \frac{a+2b \mp \sqrt{a^2+8bc}}{2(a+b-2c)}, \tag{A5}$$

where we denoted $a=f(\alpha-S)$, $b=f(\alpha-P)$, and $c=f(\alpha-R_1)$. Since $a+b\geq 2c$, and f is nondecreasing we have $\bar{\mu}_3^- < 1$ and $\bar{\mu}_3^+ > 1$, and Eq. (A1) follows by direct calculation, noticing that $a\geq b\geq c$, and identifying $\bar{\mu}_3$ with $\bar{\mu}_3^-$.

Lemma 2. Let us consider fixed aspiration level $\bar{\alpha}$ such that $\bar{\alpha}\leq R_i$ for all i except $i=N-1$, and assume that $T_N<\bar{\alpha}\leq T_i$ for all i except $i=N$. If $\bar{\alpha}$ satisfies

$$P < \bar{\alpha} \leq \min_{i\neq N-1, j\neq N}\{R_i, T_j\}, \tag{A6}$$

then the asymptotic cooperation level does not depend on the order N of the PD game.

Proof. For the aspiration level $\bar{\alpha}$ Eq. (6) takes the form

$$\dot{\mu} = -\mu(1-\mu)^{N-1} f(\bar{\alpha}-R_{N-1}) + (1-\mu)^N f(\bar{\alpha}-T_N), \tag{A7}$$

which has two fixed points, both independent of N.

APPENDIX B

We give details of proof of Theorem 1 for $N=3$. Take the first ordering

$$T_1 > R_0 > T_2 > R_1 > T_3 > R_2 \tag{B1}$$

which implies [cf. formulas (1) and (2) for $N=3$]

$$2R > T+P, \quad R+S > 2P. \tag{B2}$$

We look for solutions of Eq. (11) in the interval (0,1). Ordering (B1) defines the partition of the real axis into seven intervals.

(1) For $\alpha\in(T_1,+\infty)$ Eq. (11) has the solution $\mu=1/2$, for which the assumed consistency condition for α: $\alpha>T_1$, reads (note that in the stationary state $\alpha=2[\mu^2R+\mu(1-\mu)(S+T)+(1-\mu)^2P]$): $R+S+T+P>4T$, contradicting the 2-PD inequalities.

(2) For $\alpha\in(R_0,T_1]$ Eq. (11) reads $\mu^3-\mu^2-2\mu+1=0$, with the solution $\mu\approx 4/9$, for which the consistency conditions for α reduce approximately to $20T+20S+25P>65R$. The latter inequality cannot be satisfied together with the inequality $2R>T+P$ in Eq. (B2).

(3) For $\alpha\in(T_2,R_0]$ Eq. (11) has the solution $\mu=1/2$, for which, from Eq. (12) $\alpha=(R+S+T+P)/2<T_2$, a contradiction.

(4) For $\alpha\in(R_1,T_2]$ Eq. (11) has the solution $\mu=1/3$, for which the consistency conditions for α reduce to $4T+8P>7R+5S$. Since this consistency condition is satisfied, $\mu^*=1/3$ is the unique cooperation level.

(5) For $\alpha\in(R_1,T_2]$ Eq. (11) has the solution $\mu=1/2$, for which the consistency condition $\alpha=(R+S+T+P)/2\leq T_2$ is not satisfied.

(6) and (7). For $\alpha\leq R_1$ Eq. (11) has no solutions in (0,1).

We demonstrated that for ordering (B1) there is at most one cooperation level ($\mu^*=1/3$). In the same way we proof the existence of at most one cooperation level for $N=3$ for the other three orderings of the 3-PD payoffs. For $N=2$ the proof is analogous and will be not presented here.

APPENDIX C

Example 2: Details

For $N=5$ and the payoff matrix $[1.5,1,0,-0.4]$ formulas (1) and (2) give the following ordering of the payoffs in the 5-PD:

$$T_1 > T_2 > R_0 > T_3 > R_1 > T_4 > R_2 > T_5 > R_3 > R_4.$$

We proceed as in Example 1. Let (μ^*,α^*) be a stationary state of Eqs. (6)–(10); α^* belongs to an interval determined by the above ordering. For this interval we write down the explicit form of the function $F=F(\mu^*,\Pi(\mu^*))$, which for the Heaviside switch function is a polynomial of at most fifth degree, and determine its zeros $\mu^*\in(0,1)$. If α^* calculated from the consistency condition $\alpha^*=\Pi(\mu^*)$ belongs to the chosen interval, then μ^* is the cooperation level. We apply this procedure subsequently to all the intervals determined by the above ordering. For each interval Eq. (11) has different analytical form.

For the first three intervals, $-1.6=R_4\geq\alpha>-\infty$, $-0.2=R_3\geq\alpha>R_4$, $4.5=T_5\geq\alpha>R_3$ Eq. (11) has no solutions in (0,1).

For the next interval $1.2=R_2\geq\alpha>T_5$ Eq. (11) takes the form $(1-\mu)^4(1-2\mu)-4(1-\mu)^3\mu^2=0$, with the cooperation level solution $\mu_0=(-3+\sqrt{17})/4\in(0,1)$. However, the consistency condition $R_2\geq\alpha$ with α calculated from stationary-state relation (12) written for $N=5$ and the payoff matrix $[1.5,1,0,-0.4]$ takes the form $\alpha=4\mu_0(1.1-0.1\mu_0)\leq 1.2$, which is not true (the latter inequality leads to rather "subtle" false result $425\leq 424.36$, which shows the sensitivity of the procedure to the changes of the parameters of the model).

For the interval $1.5=T_4\geq\alpha>R_2$ Eq. (11) has the solution $\mu_0=1/4$, for which $\alpha_0=1.075<R_2$, a contradiction.

For the interval $2.6=R_1\geq\alpha>T_4$ Eq. (11) has the solution $\mu_0=1/\sqrt{7}$, for which $\alpha_0\approx 1.61$, consistently with the assumed interval of α. Thus, this interval gives the partial cooperation level μ_0.

For the interval $3=T_3\geq\alpha>R_1$ Eq. (11) reads $-4\mu^4+7\mu^3-7\mu^2-\mu+1=0$; for $4=R_0\geq\alpha>T_3$ Eq. (11) takes the form $2\mu^4-5\mu^3-\mu^2-\mu+1=0$. In both cases the (unique) solution in (0,1) leads to the value of α which does not belong to its *a priori* assumed interval.

Finally, for $\alpha>R_0=4$ there are no solutions since the relation $\alpha=4\mu_0(1.1-0.1\mu_0)$ implies $\alpha=4$ as the maximal value of α. In this way we proved the existence of the unique cooperation level $\mu_0=1/\sqrt{7}$.

Example 3

For $N=3$, and the 2-PD payoff matrix $[T,1,0,1-T]$, $T=2$, from Theorem 2 we calculate: $T=2$: $T_1=4$, $T_2=2$, $T_3=0$, $R_0=2$, $R_1=0$, and $R_2=-2$, and, from Eq. (12), $\alpha=2\mu\in(T_3,R_0]$. For the switch function f_1 with $p=1$ Eq. (11) takes the form: $-\mu(1-\mu)^2+(1-2\mu-\mu^2)(1-2\mu)\alpha=0$, which, with $\alpha=2\mu$ has the solution $\mu=(-3+\sqrt{17})/4$.

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